

## Differential Eigenvalue Problems in Which the Parameter Appears Nonlinearly

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Several methods are examined for determining the eigenvalues of a system of equations in which the parameter appears nonlinearly. The equations are the result of the discretization of differential eigenvalue problems using a finite Chebyshev series. Two global methods are considered which determine the spectrum of eigenvalues without an initial estimate. A local iteration scheme with cubic convergence is presented. Calculations are performed for a model second order differential problem and the Orr-Sommerfeld problem for plane Poiseuille flow.

### 1. INTRODUCTION

This paper is concerned with the approximate solution of eigenvalue problems for ordinary differential equations of the type

$$\sum_{i=0}^p \alpha^{p-i} L_i \phi = 0, \quad (1)$$

where  $L_i$  are linear differential operators,  $\phi$  is a complex variable and the parameter  $\alpha$  is an eigenvalue. Such equations frequently arise in many fields of application. The Orr-Sommerfeld problem of hydrodynamic stability theory, which is considered in this paper, is of this type.

The approximate solution of (1) for  $p = 1$  has been studied extensively and many solution techniques have been developed. Two matrices may be formed from the discretization of the operators  $L_0$  and  $L_1$ . This results in a generalized eigenvalue problem for which many globally convergent solution algorithms have been constructed [1, 2]. When  $p$  is greater than one the problem is said to be nonlinear in the parameter  $\alpha$ . Such a case arises in the study of the linear spatial stability of laminar flows. The Orr-Sommerfeld equation is obtained as a first order perturbation of the Navier-Stokes equations. This equation contains the parameter  $\alpha$  to the power four. For spatial stability calculations, the sign of the imaginary part of  $\alpha$  determines

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whether the perturbations will grow or decay in space. Exact solutions to the Orr–Sommerfeld problem exist only in special cases. In general, an approximate solution is sought.

Most previous solution techniques for the Orr–Sommerfeld problem have used local iterations to determine an eigenvalue. A commonly used method is the use of shooting with orthonormalization [3], in which a first guess is made for the eigenvalue and then integration is performed over the domain. This is repeated until the boundary conditions at the end of the domain match the computed solution. The difficulties with such methods are that a good first guess is required to assure convergence and the determination of eigensolutions other than the least stable may be difficult.

This paper develops a globally convergent method of solution to the nonlinear eigenvalue problem. An accurate discretization is used to convert the operators to matrices. Then, with no initial approximation, all the eigenvalues of a particular discretization or a specified subset of the eigenvalue spectrum may be determined.

In Section 2 the method of solution is described. To develop the methodologies the technique is applied to a simple model problem with a known exact solution. In Section 3 the techniques are applied directly to the Orr–Sommerfeld problem for plane Poiseuille flow. The advantages of a cubically convergent local method are also described. In Section 4 some particular problems, which arise in practice, such as infinite eigenvalues, singular matrices and spurious eigenvalues are discussed. The application of Newton's method for determining the root matrices of matrix polynomials is also described.

## 2. OUTLINE OF THE METHOD

The approximate solution of (1) requires two steps. First the operators  $L_i$  are converted to matrices. Clearly a discretization is desired which yields the most accurate solution for the minimum order of the matrix. In the present method the discretization of (1) involves the expansion of  $\phi$  in a finite Chebyshev series. If the coefficients in the linear operators are infinitely differentiable then a finite Chebyshev series for  $\phi$  is significantly more accurate than either a finite-difference approximation or an expansion in other orthogonal polynomials [4]. In the second step the resulting nonlinear eigenvalue problem is solved. Some recent developments in linear algebra [5], particularly the general properties of matrix polynomials, are used. Both linearization and factorization are used in solving the nonlinear eigenvalue problem.

In describing the solution techniques it is helpful to consider a simple model problem. Consider the following differential problem in the interval  $x \in [-1, 1]$

$$\frac{d^2\phi}{dx^2} - 2\alpha\omega \frac{d\phi}{dx} + \alpha^2\phi = 0 \quad (2)$$

$$\phi(1) = \phi(-1) = 0. \quad (3)$$

For given values of  $\omega$  the eigenvalues  $\alpha$ , and eigenfunctions  $\phi(\alpha)$ , are to be determined. The exact solution is given by

$$4\alpha^2(1 - \omega^2) = n^2\pi^2, \quad n = 1, 2, \dots \tag{4}$$

$$\phi(x) = \begin{cases} Ae^{\omega\alpha x} \cos(n\pi x/2): & n \text{ odd} \\ Ae^{\omega\alpha x} \sin(n\pi x/2): & n \text{ even.} \end{cases} \tag{5}$$

An approximate solution is obtained by expanding  $\phi$  in a finite Chebyshev series,

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^N a_n T_n(x), \tag{6}$$

where  $T_n(x)$  denotes an  $n$ th order Chebyshev polynomial of the first kind. The integral of a Chebyshev series takes a more convenient form than its derivative. Thus, (2) is integrated to obtain,

$$\phi(x) - 2\alpha\omega \int \phi + \alpha^2 \iint \phi + b_1 x + b_2 = 0. \tag{7}$$

The series (6) is substituted into (7) and the integrations performed making use of the Chebyshev integral relation,

$$\int T_n(x) dx = \begin{cases} \frac{T_{n+1}(x)}{2(n+1)} - \frac{T_{n-1}(x)}{2(n-1)}, & n \geq 2 \\ \frac{1}{4}[T_0(x) + T_2(x)], & n = 1 \\ T_1(x), & n = 0. \end{cases} \tag{8}$$

Equating coefficients of the Chebyshev polynomials of equal order leads to a system of linear equations. To obtain a solution to the finite system of equations the Lanczos tau method [5] is used. This involves the addition of a term to the right hand side of (7) of the form

$$E_2(x) = \tau_1 T_{N+1}(x) + \tau_2 T_{N+2}(x). \tag{9}$$

The resulting solution is the exact solution to (7) perturbed by (9). This analysis leads to  $N + 3$  equations and two boundary conditions for the  $N + 5$  unknowns  $b_1, b_2, a_0, a_1, \dots, a_N, \tau_1, \tau_2$ . The equations involving the coefficients of  $T_0(x)$  and  $T_1(x)$  only serve to determine the constants of integration  $b_1$  and  $b_2$  and will be discarded. The tau terms need not be calculated explicitly either. The remaining  $N - 1$  equations are, for  $n = 2, 3, \dots, N$ ,

$$a_n - \frac{\alpha\omega}{n} (a_{n-1} - a_{n+1}) + \frac{\alpha^2}{4n} \left( \frac{a_{n-2}}{n-1} - \frac{2na_n}{n^2-1} + \frac{a_{n+2}}{n+1} \right) = 0. \tag{10}$$

The remaining two equations for the  $N + 1$  unknowns  $a_0, a_1, \dots, a_N$  are obtained from the boundary conditions,

$$\phi(\pm 1) = \frac{a_0}{2} + \sum_{n=1}^N (\pm 1)^n a_n = 0. \tag{11}$$

The equation involving the coefficients of  $T_{N+1}(x)$  and  $T_{N+2}(x)$  may be used to ensure a nontrivial solution by setting,

$$\tau_1 = \frac{\alpha^2 a_{N-1}}{4N(N+1)} - \frac{\alpha \omega a_N}{N+1}, \tag{12}$$

and

$$\tau_2 = \frac{\alpha^2 a_N}{4(N+1)(N+2)}. \tag{13}$$

Thus, the tau terms need not be evaluated although they do provide some estimate of the error in the approximate solution. It is interesting to note that in this example the error is largely dependent on  $\alpha/N$ . Thus, a large number of terms would be necessary for the accurate calculation of those eigenvalues with large amplitudes, which are usually termed the higher modes. This is not to be confused with the order of accuracy of the solution. The order of accuracy of the Chebyshev polynomial solutions,  $f(N) = O(N^{-p})$ , carries with it the implication; as  $N \rightarrow \infty$ . In this paper, the solutions are infinitely differentiable, therefore, the convergence is faster than any finite power of  $p$ . Therefore, they are referred to as “infinite order accurate.” However, this yields no estimate of the error at particular values of  $N$ . In the above, we note the dependence of this particular error on  $\alpha/N$ .

The resulting homogeneous system of equations is an eigenvalue problem, nonlinear in the parameter  $\alpha$ , in which the Chebyshev series coefficients form the eigenvector.

$$\mathbf{D}_2(\alpha) \mathbf{a} = 0. \tag{14}$$

The matrix  $\mathbf{D}_2(\alpha)$  is a lambda matrix of degree two [7], which may be expressed as a scalar polynomial with matrix coefficients,

$$\mathbf{D}_2(\alpha) = \mathbf{C}_0 \alpha^2 + \mathbf{C}_1 \alpha + \mathbf{C}_2, \tag{15}$$

where the matrices  $\mathbf{C}_0, \mathbf{C}_1$  and  $\mathbf{C}_2$  are square matrices of order  $N + 1$  with, in general complex elements. Since, in this problem, the boundary conditions are independent of the parameter  $\alpha$ , the matrices  $\mathbf{C}_0$  and  $\mathbf{C}_1$  will have a rank less than or equal to  $N - 1$ .

The eigenvalues of (14) are the roots of the latent equation  $\det \mathbf{D}_2(\alpha) = 0$ . This is a scalar polynomial equation with degree that does not exceed  $2(N + 1)$ . When  $\mathbf{C}_0$  is singular the latent equation has degree less than  $2(N + 1)$  so that the problem has fewer than  $2(N + 1)$  eigenvalues. If  $\mathbf{C}_2$  is non-singular the missing eigenvalues

correspond to zero eigenvalues of the reciprocal problem:  $\det \mathbf{D}_2(\mu) = 0$ , with  $\mathbf{D}_2(\mu) = \mathbf{C}_2\mu^2 + \mathbf{C}_1\mu + \mathbf{C}_0$ , and  $\mu = 1/\alpha$ . Thus, the missing eigenvalues are referred to as "infinite eigenvalues." Similarly, if  $\mathbf{C}_0$  is nearly singular, some very large eigenvalues will appear. However, if the other eigenvalues are well-conditioned, they are not affected by these large eigenvalues. This is discussed further in Section 4.

The matrices  $\mathbf{C}_0$ ,  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are constructed by setting Eqs. (10) in the first  $N - 1$  rows and the boundary conditions (11) in the last two rows. Then  $\mathbf{C}_0$ ,  $\mathbf{C}_1$  and  $\mathbf{C}_2$  will be upper triangular with the lowest two rows of  $\mathbf{C}_2$  containing the boundary condition entries. The elements in the lowest two rows of  $\mathbf{C}_2$  are eliminated except for the  $(N, N)$ ,  $(N, N + 1)$  and  $(N + 1, N + 1)$  entries with column operations. This preserves the zeroes in the lowest two rows of  $\mathbf{C}_0$  and  $\mathbf{C}_1$ . If the  $(N, N)$  and  $(N + 1, N + 1)$  entries are non-zero (which will always be the case for linearly independent boundary conditions) the problem is clearly reduced to an  $(N - 1)$  by  $(N - 1)$  nonlinear eigenvalue problem. In most cases, this elimination of the boundary conditions is sufficient to rid the problem of infinite eigenvalues and make  $\mathbf{C}_0$  non-singular. However, this is not always the case and in Section 4 such a situation is described.

To find the eigenvalues of the nonlinear matrix eigenvalue problem two methods are considered which are globally convergent. The first determines the entire eigenvalue spectrum without an initial guess. The second method determines a specified subset of the eigenvalue spectrum, also without an initial guess.

### 2.1 The Linear Companion Matrix Method

A companion matrix for the lambda matrix may be formed in the same manner as for a scalar polynomial. The companion matrix is a linearization of the lambda matrix and has a larger order. In subsequent discussion the order of the lambda matrix will be taken as  $m$ . After the column operations of the previous section  $m = N - 1$ . For a lambda matrix of degree  $l$  the order of the companion matrix is  $lm$ . Using the transformation  $\mathbf{a}_1 = \alpha \mathbf{a}$  one such companion matrix may be constructed for Eq. (15),

$$\left\{ \begin{bmatrix} -\mathbf{C}_1 & -\mathbf{C}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix} - \alpha \begin{bmatrix} \mathbf{C}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right\} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a} \end{Bmatrix} = \{\mathbf{0}\}. \tag{16}$$

This is a generalized eigenvalue problem and many globally convergent algorithms exist for its solution [1, 2]. However, if  $\mathbf{C}_0$  is non-singular it is more efficient to convert (16) to the algebraic eigenvalue problem of finding the eigenvalues of the matrix  $\mathbf{A}$ , given by

$$\mathbf{A} = \begin{bmatrix} -\mathbf{C}_0^{-1}\mathbf{C}_1 & -\mathbf{C}_0^{-1}\mathbf{C}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{17}$$

These eigenvalues may be obtained very efficiently and accurately using the *QR* algorithm.

In the model problem (15) the matrix  $\mathbf{A}$  is of order  $2N - 2$ . Calculations have been performed for  $\omega = 1 + i$  and various values of  $N$ . The error in the first three eigenvalues is shown in Table I. The errors given in Table I are the magnitudes of the differences between the computed eigenvalues and the exact solutions given by (4) with  $n = 1, 2, 3$  and  $\omega = 1 + i$ .

### 2.2 Factorization of the Lambda Matrix

The increase in the order of the matrix eigenvalue problem that occurred in the previous section may be avoided by factorizing the polynomial. If the lambda matrix is of order two, this factorization results in two linear problems. The matrix analogue of synthetic division is known as the Generalized Bézout Theorem [8, p. 81]. In the matrix case the division must be defined as either left or right division, since matrices are generally non-commutative. Using the Bézout theorem, the lambda matrix of degree two, Eq. (14), has the factorization

$$\mathbf{D}_2(\alpha) = [\mathbf{C}_0\alpha + \mathbf{C}_0\mathbf{Y} + \mathbf{C}_1][\mathbf{I}\alpha - \mathbf{Y}] + \mathbf{C}_0\mathbf{Y}^2 + \mathbf{C}_1\mathbf{Y} + \mathbf{C}_2. \quad (18)$$

$\mathbf{Y}$  is referred to as the right solvent. An equivalent left solvent could also be derived. From the factorization of  $\mathbf{D}_2(\alpha)$ , it is clear that if a root matrix of the matrix polynomial,

$$\mathbf{C}_0\mathbf{Y}^2 + \mathbf{C}_1\mathbf{Y} + \mathbf{C}_2 = 0, \quad (19)$$

is found it will contain  $m$  eigenvalues of  $\mathbf{D}_2(\alpha)$ . Similarly, the right eigenvectors of  $\mathbf{Y}$  coincide with the right latent vectors of  $\mathbf{D}_2(\alpha)$ . However, given the latent roots of a lambda matrix,  $m$  of these cannot be arbitrarily assigned to the eigenvalues of a solvent matrix, since the associated latent vectors must also be eigenvectors of the solvent matrix. The eigenvectors of a matrix associated with unequal eigenvalues are independent. This is not necessarily true for the latent vectors of a lambda matrix with degree greater than one. Thus, the unequal latent roots, which are also eigenvalues of some solvent matrix, must have associated linearly independent latent vectors. Although examples may be contrived which do not meet this requirement no

TABLE I  
Effect of the Order of the Chebyshev Polynomials,  $N$ , on the Eigenvalues Obtained  
Using the Companion Matrix Method:  $\omega = 1 + i$

Polynomial Order $N + 1$	$\alpha_1$ error	$\alpha_2$ error	$\alpha_3$ error
8	1.32(-4)	2.47(-2)	6.83(-01)
10	1.39(-06)	1.61(-3)	8.15(-02)
16	3.28(-09)	7.26(-07)	3.53(-04)
20	6.24(-15)	9.00(-09)	5.01(-07)

unequal eigenvalues with linearly dependent eigenvectors have been encountered in the present computations.

The matrix polynomial (19) may be solved in a manner analogous to the scalar equivalent. Though, in general, direct application of scalar methods is hampered by the noncommutivity of the matrices. For example, the quadratic formula fails unless  $C_1$  and  $Y$  commute, which is unlikely. An iterative method, developed by Bernoulli, for finding the dominant root of a scalar polynomial, can be generalized to find the root matrices of a matrix polynomial [5, p. 126]. The matrix Bernoulli method may be used to compute a dominant solvent or a minimal<sup>1</sup> solvent. A solvent is said to be minimal (dominant) if every eigenvalue in it has an absolute value less (greater) than all eigenvalues of  $D_2(\alpha)$  not in the minimal (dominant) solvent.

As in the scalar case, a matrix difference equation is constructed for the sequence  $U_n$ ,

$$C_0 U_n + C_1 U_{n+1} + C_2 U_{n+2} = 0, \quad (20)$$

with initial conditions  $U_0 = 0$ ,  $U_1 = I$ . Then

$$\lim_{n \rightarrow \infty} U_n U_{n+1}^{-1} = Y, \quad (21)$$

where  $Y$  is the minimal solvent, under the conditions that (a) the eigenvalues of the lambda matrix can be divided into two non-intersecting groups such that,

$$|\alpha_1| \leq |\alpha_2| \leq \dots \leq |\alpha_m| < |\alpha_{m+1}| \leq \dots \leq |\alpha_{2m}|, \quad (22)$$

and (b) independent eigenvectors of  $D_2(\alpha)$  exist for the first  $m$  eigenvalues  $\alpha_1, \alpha_2, \dots, \alpha_m$ . The equivalent difference equation for the dominant solvent may also be derived [5, p. 126]. Considering the form of (21) the sequence  $Y_n$  is obtained from the relation

$$(C_0 Y_n + C_1) Y_{n+1} = -C_2, \quad (23)$$

with  $Y_0 = 0$ . This may be extended to any order of matrix polynomial, *mutatis mutandis*. In the next section a fourth order matrix polynomial is considered. The subroutine for Bernoulli iteration is quite easy to implement, requiring about twenty lines of code, including a matrix multiply and inverse routine. On convergence, the eigenvalues of  $Y$  are computed using the *QR* algorithm. For the present calculations, the number of iterations has been fixed but the process could be automated by defining a residual matrix,

$$R_n = [C_0 Y_n + C_1] Y_n + C_2, \quad (24)$$

<sup>1</sup> There appears to be no term in general use for this solvent. The adjective minimal means the "least possible" or "smallest; very small."

and iterating until the norm of  $\mathbf{R}_n$  is less than some specified tolerance. The first three eigenvalues of the model problem have been computed for  $\omega = 1 + i$ . The results are given in Table II and may be compared with the results in Table I which were obtained using the linear companion matrix method.

2.3 Local Iteration and Eigenvectors

Once a sufficiently good guess is available for a single eigenvalue the root may be calculated using a locally convergent algorithm. The very efficient methods of Lancaster [9] may be used for final refinement of selected eigenvalues and computation of the associated eigenvectors.

Newton's method may be applied to find the zeroes of  $\det \mathbf{D}(\alpha)$ . However, the determinant and its derivative with respect to  $\alpha$  need not be determined explicitly if the trace theorem of Davidenko [10] is used. The iterative formula may be written

$$\alpha_{k+1} = \alpha_k - 1/f(\alpha_k), \quad k = 0, 1, 2, \dots, \tag{25a}$$

where

$$f(\alpha_k) = T_r\{\mathbf{D}^{-1}(\alpha_k) \mathbf{D}^{(1)}(\alpha_k)\}. \tag{25b}$$

$T_r\{\mathbf{A}\}$  denotes the trace of  $\mathbf{A}$ ,  $\mathbf{D}^{-1}$  is the inverse of  $\mathbf{D}$  and  $\mathbf{D}^{(1)}$  denotes the derivative of  $\mathbf{D}$  with respect to  $\alpha$ . This algorithm (25) is quadratically convergent. However, a refinement of this method, derived by Lancaster [9], is cubically convergent. The iterative formula is

$$\alpha_{k+1} = \alpha_k - 2f(\alpha_k)/\{|f(\alpha_k)|^2 - f^{(1)}(\alpha_k)\}, \quad k = 0, 1, 2, \dots, \tag{26}$$

where  $f(\alpha_k)$  is given by (25) and

$$f^{(1)} = T_r\{\mathbf{D}^{-1}\mathbf{D}^{(2)} - [\mathbf{D}^{-1}\mathbf{D}^{(1)}]^2\}. \tag{27}$$

It should be noted that in computing  $f^{(1)}(\alpha_k)$  the inversion required to calculate  $f(\alpha_k)$  may be used. Further effort is required to compute the entire product  $\mathbf{D}^{-1}\mathbf{D}^{(1)}$ .

TABLE II  
Effect of the Order of the Chebyshev Polynomials,  $N$ , on the Eigenvalues Obtained  
Using Matrix Factorization:  $\omega = 1 + i$

Polynomial Order $N + 1$	$\alpha_1$ error	$\alpha_3$ error	$\alpha_5$ error	Number of Iterations
8	1.32(-04)	2.47(-02)	6.81(-01)	10
10	3.58(-06)	3.74(-03)	1.23(-01)	10
16	4.76(-08)	7.58(-07)	3.58(-04)	10
20	1.40(-10)	7.05(-08)	3.40(-06)	10



To compute single eigenvectors the inverse iteration is used

$$\mathbf{D}(\alpha) \mathbf{x}^{k+1} = \sigma \mathbf{x}^k, \quad (28)$$

where  $\sigma$  is a scaling factor. Even when  $\mathbf{D}(\alpha)$  is nearly singular this procedure is very effective. Convergence is usually obtained in two iterations using an initial guess,  $\mathbf{x}^0 = [1, 1, \dots, 1]^T$ .

Since the inverse of  $\mathbf{D}(\alpha)$  has been computed in the local iteration for the eigenvalue the eigenvector calculation may be coupled to this procedure. Once the eigenvalue has been computed to sufficient accuracy usually only two matrix-vector multiplies, about  $N^2$  operations, are needed to compute the corresponding eigenvector. For single eigenvalues, it has been found to be efficient to determine a first guess using the Bernoulli method, Section 2.2, using a low order polynomial and then refining the value using the local method with a higher order polynomial.

For example, with  $\omega = 1 + i$  in the model problem and  $N = 7$ , 5 iterations with the Bernoulli method give the first eigenvalue as

$$\alpha_1 = (0.893805, 0.552098),$$

with

$$\alpha_1 \text{ (abs. error)} = 2.83(-04).$$

Using this as the first guess, and with  $N = 15$ , after 2 iterations the refined eigenvalue obtained using the local iteration is

$$\alpha_1 = (0.89357024, 0.55225678)$$

$$\alpha_1 \text{ (abs. error)} = 8.3 (-12).$$

### 3. THE ORR-SOMMERFELD EQUATION

The Orr-Sommerfeld equation is obtained as a linearization of the incompressible Navier-Stokes equations in which the perturbations in velocity and pressure are assumed to take the form (for two-dimensional disturbances)

$$\Phi(x, y, t) = \phi(y) \exp[i(ax - \omega t)]. \quad (29)$$

$\alpha$  is a wavenumber and  $\omega$  is a radian frequency. The basic velocity profile, given by  $U(y)$ , is assumed to be known and for a given Reynolds number  $R$  the Orr-Sommerfeld equation may be written

$$\left[ \left( \frac{d^2}{dy^2} - \alpha^2 \right)^2 - iR \right] (\alpha U - \omega) \left( \frac{d^2}{dy^2} - \alpha^2 \right) - \alpha U'' \left[ \right] \phi = 0. \quad (30)$$

The case of plane channel flow will be considered for which  $U = (1 - y^2)$  and the side walls are at  $y = \pm 1$ . The boundary conditions are then

$$\phi(y) = 0, \quad \phi'(y) = 0 \quad \text{at } y = \pm 1 \quad (31)$$

for a given real value of Reynolds number only certain combinations of values of  $\alpha$  and  $\omega$  will permit  $\phi$  to satisfy the boundary conditions. In general  $\alpha$  and  $\omega$  are complex, however, two special cases are of interest. If  $\alpha$  is fixed and real and  $\omega$  is the complex eigenvalue, the disturbances are periodic in  $x$  and grow or decay with time depending on the sign of the imaginary part of  $\omega$ . This is referred to as temporal stability. In this case, the parameter  $\omega$  appears linearly in the equation. Calculations for this case, using Chebyshev polynomials, have been performed by Orszag [11]. In most real flows, the disturbances are periodic in time (real  $\omega$ ) and grow or decay exponentially with distance  $x$  (complex  $\alpha$ ). Calculations of this type are termed spatial stability calculations. In this case, the parameter  $\alpha$  appears nonlinearly in the equation to the fourth power. Approximate transformations do exist [12] between the temporal and spatial eigenvalues. However, this section is concerned with the direct calculation of the spectrum of spatial eigenvalues using the techniques described in Section 2.

Equation (30) is integrated indefinitely four times, giving

$$\begin{aligned} \phi - (i\omega R - 2\alpha^2) \iint \phi - i\alpha R \iint \iint U\phi + 2i\alpha R \iint \iint \iint U'\phi \\ + (\alpha^4 - i\alpha^2\omega R) \iiiii \phi + i\alpha^3 R \iiiii U\phi + \frac{b_1 y^3}{6} + \frac{b_2 y^2}{2} + b_3 y + b_4 = 0. \end{aligned} \quad (32)$$

The calculations may be performed for either symmetric or antisymmetric eigenmodes since they are uncoupled [13]. The symmetric modes, which are less stable, will be considered.  $\phi(y)$  is approximated by a finite, even Chebyshev series,

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^N a_n T_{2n}(y). \quad (33)$$

The basic flow,  $U(y)$ , may be written in terms of Chebyshev polynomials as

$$U(y) = \frac{1}{2} [1 - T_2(y)]. \quad (34)$$

The Chebyshev product formula

$$2T_m(y) T_n(y) = T_{m+n}(y) + T_{m-n}(y) \quad (35)$$

is used, as well as (8), and (33) and (34) are substituted into (32) and integrated. The tau method is used to close the system of equations by adding a term to the right hand side of (32) of the form

$$E_3(y) = \tau_1 T_{2N+2}(y) + \tau_2 T_{2N+4}(y) + \tau_3 T_{2N+6}(y). \quad (36)$$

The exact solution may then be found to this perturbed equation. The coefficients of the various  $T_{2n}(y)$  are equated to zero to form a system of linear equations. In the present case, the constants of integration  $b_1$  and  $b_3$  are zero and the  $T_0(y)$  and  $T_2(y)$  equations are discarded since there is no need to calculate  $b_2$  and  $b_4$ . The tau terms need not be calculated explicitly so that a system of  $N - 1$  equations for the  $(N + 1)$  unknowns  $a_0, a_1, \dots, a_N$ , results. For  $n = 2, 3, \dots, N$  these equations have the form

$$\alpha^4 \tilde{B}_n + i\alpha^3 R(\tilde{B}_n - \tilde{E}_n)/2 - \alpha^2(2\tilde{A}_n + i\omega R\tilde{B}_n) - i\alpha R(\tilde{B}_n - \tilde{D}_n + 8\tilde{C}_n)/2 + (a_n + i\omega R\tilde{A}_n) = 0. \tag{37}$$

The coefficients  $\tilde{A}_n, \tilde{B}_n$ , etc., are given in the Appendix. These coefficients need not be derived for each velocity profile. In general, if

$$U(y) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n T_n(y), \tag{38}$$

and

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^N a_n T_n(y), \tag{39}$$

then

$$U(y)\phi(y) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n T_n(y), \tag{40}$$

where

$$d_n = \frac{b_n a_0}{2} + \frac{1}{2} \sum_{m=1}^N (b_{|m-n|} + b_{m+n}) a_m, \quad n \geq 0. \tag{41}$$

Equation (40) may then be truncated as desired and integrated using (8). Since an even solution is sought, two of the boundary conditions are unnecessary. The remaining boundary conditions give

$$\frac{a_0}{2} + \sum_{n=1}^N a_n = 0 \tag{42}$$

and

$$\sum_{n=1}^N n^2 a_n = 0. \tag{43}$$

Equations (37), (42) and (43) give a system of  $N + 1$  homogeneous equations in which the parameter  $\alpha$  appears to the fourth power

$$\mathbf{D}_4(\alpha) \mathbf{a} = 0, \tag{44}$$

where

$$\mathbf{D}_4(\alpha) = \mathbf{C}_0 \alpha^4 + \mathbf{C}_1 \alpha^3 + \mathbf{C}_2 \alpha^2 + \mathbf{C}_3 \alpha + \mathbf{C}_4 \tag{45}$$

and  $C_i, i = 0, 1, \dots, 4$  are matrices of order  $N + 1$  with complex elements. The lowest two rows of  $C_0, C_1, C_2$  and  $C_3$  are zero since the boundary conditions are independent of  $\alpha$ . As before, the lowest two rows of  $C_4$  may be eliminated using column operations except for a  $2 \times 2$  submatrix which is upper triangular. Since (42) and (43) are linearly independent this submatrix is nonsingular and the problem is reduced to a quartic eigenvalue problem of order  $N - 1$ .

From (37) it can be seen that all the coefficient matrices  $C_i$ , except  $C_0$ , depend on the Reynolds number which may give their elements large magnitude. However, the elements of  $C_0$  all have a magnitude less than one. For this reason,  $C_0$  has a small determinant though it may still be inverted. The consequences of this are discussed in Section 4.

The eigenvalues of the quartic eigenvalue problem (44) may be obtained using the methods developed in Section 2.

### 3.1 The Companion Matrix Method

One companion matrix that may be formed to give a linear generalized eigenvalue problem is

$$\left\{ \begin{bmatrix} -C_1 & -C_2 & -C_3 & -C_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} - \alpha \begin{bmatrix} C_0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \right\} \begin{Bmatrix} \alpha^3 \mathbf{a} \\ \alpha^2 \mathbf{a} \\ \alpha \mathbf{a} \\ \mathbf{a} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (46)$$

where  $\mathbf{a}$  is the right latent vector of  $D_4(\alpha)$ . The large order of the system makes this

Fobenius companion matrix may be formed instead,

$$A = \begin{bmatrix} -C_0^{-1}C_1 & -C_0^{-1}C_2 & -C_0^{-1}C_3 & -C_0^{-1}C_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}. \quad (47)$$

Matrix  $A$  is of order  $4m$ , where  $m$  is the order of  $D_4(\alpha)$ . The linear companion matrices are not unique. A new companion matrix may be formed by any row and column permutation. For example,

$$A_1 = \begin{bmatrix} -C_0^{-1}C_1 & -C_0^{-1}C_3 & -C_0^{-1}C_2 & -C_0^{-1}C_4 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \quad (48)$$

is also a companion matrix to  $D_4(\alpha)$ .  $A_1$  is also a companion matrix to a lambda matrix of degree two.

In either form (47) or (48) the companion matrix may be balanced and the  $QR$  algorithm used to compute all the eigenvalues. Since the  $QR$  algorithm requires computation times of order  $(4m)^3$  the solution of the block companion matrix will require significantly more computer time than a linear problem of order  $m$ . However, this method is globally convergent and for matrices of reasonable order the eigenvalues may be computed efficiently and accurately with no initial guess.

The linear eigenvalue problem (temporal stability) has been solved by Orszag [11]. The critical neutral point ( $\alpha$  and  $\omega$  both real for minimum  $R$ ) was found to be at  $R = 5772$ ,  $\alpha = 1.02056$  with an eigenvalue  $\omega \simeq 0.26943$ . In the present calculations the values of  $R$  and  $\omega$  are set at 5772 and 0.26943 and the value of  $\alpha$  is calculated. Table III shows the results for several values of polynomial order.

In a related paper, Bramley and Dennis (14) have used the companion matrix method to solve a nonlinear eigenvalue problem. The differential equation they address is the limiting case of the Orr-Sommerfeld equation where the frequency is zero. The advantage of this equation is that the coefficients are real. Therefore, the resulting generalized eigenvalue problem is real, and the widely available  $QZ$  algorithm can be used. We consider here the more general case where the equation and resulting eigenvalue problem are complex.

### 3.2 Factorization of the Lambda Matrix

The lambda matrix  $D_4(\alpha)$  may also be factorized using the Bézout theorem giving,

$$D_4(\alpha) = Q_3(\alpha)(I\alpha - Y) + D_4(Y), \tag{49}$$

where  $Q_3(\alpha)$  is a lambda matrix of degree three and  $Y$  is a right solvent. The factorization is clearly equivalent to solving the fourth order matrix polynomial

$$D_4(Y) = C_0 Y^4 + C_1 Y^3 + C_2 Y^2 + C_3 Y + C_4 = 0. \tag{50}$$

TABLE III

Effect of Order of the Chebyshev Polynomials,  $N$ , on the Solution of the Orr-Sommerfeld Equation Using the Companion Matrix Method:  $R = 5772$ ,  $\omega = 0.26943$

Polynomial Order (+1) $N + 1$	$\alpha$
12	1.019519 + i7.83(-03)
16	1.020723 + i8.03(-04)
24	1.020556 + i4.61(-06)
32	1.020556 + i9.74(-07)
36	1.020556 + i9.74(-07)

A solvent  $\mathbf{Y}$  may be calculated such that it is the minimal solvent using the Bernoulli matrix difference equation

$$\mathbf{C}_0 \mathbf{U}_{n-2} + \mathbf{C}_1 \mathbf{U}_{n-1} + \mathbf{C}_2 \mathbf{U}_n + \mathbf{C}_3 \mathbf{U}_{n+1} + \mathbf{C}_4 \mathbf{U}_{n+2} = 0 \quad (51)$$

with initial values  $\mathbf{U}_{-2} = \mathbf{U}_{-1} = \mathbf{U}_0 = \mathbf{0}$ ,  $\mathbf{U}_1 = \mathbf{I}$ . The sequence  $\mathbf{U}_2, \mathbf{U}_3, \dots$ , may then be calculated. If it is assumed that (a) the eigenvalues of  $\mathbf{D}_4(\alpha)$  can be divided into two non-intersecting groups such that

$$|\alpha_1| \leq |\alpha_2| \leq \dots \leq |\alpha_m| < |\alpha_{m+1}| \leq \dots \leq |\alpha_{4m}| \quad (52)$$

and (b) linearly independent eigenvectors exist for the first  $m$  eigenvalues  $\alpha_1, \alpha_2, \dots, \alpha_m$ , then

$$\lim_{n \rightarrow \infty} \mathbf{U}_n \mathbf{U}_{n+1}^{-1} = \mathbf{Y}. \quad (53)$$

The sequence  $\mathbf{Y}_n = \mathbf{U}_n \mathbf{U}_{n+1}^{-1}$  may be computed from the iterative formula

$$[[[\mathbf{C}_0 \mathbf{Y}_{n-2} + \mathbf{C}_1] \mathbf{Y}_{n-1} + \mathbf{C}_2] \mathbf{Y}_n + \mathbf{C}_3] \mathbf{Y}_{n+1} = -\mathbf{C}_4 \quad (54)$$

with initial values  $\mathbf{Y}_{-2} = \mathbf{Y}_{-1} = \mathbf{Y}_0 = \mathbf{0}$ . Note that  $\mathbf{C}_0$  need not be nonsingular. As in Section 2 the implementation of (54) is straightforward requiring only matrix multiply and inverse routines.

The results of the present calculations will be compared with the numerical study of the linear spatial stability of plane Poiseuille flow by Itoh [15]. Nishioka *et al.* [16] showed that Itoh's calculations compared favorably with their experimental results. Itoh used a power series discretization and a local iteration to determine the eigenvalues. The present calculations for the first two modes for a Reynolds number of 6000 are shown in Figs. 1 and 2. The first mode corresponds exactly with the calculations of Itoh. The second mode, which is always decaying, was not selected in Itoh's calculations. However, at low and high frequencies this mode is the least stable. The second mode has a phase velocity,  $\omega/\alpha_r$ , nearly equal to unity and is thus

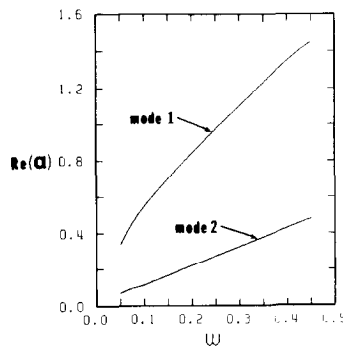


FIG. 1. Real part of eigenvalue for first two modes for  $R = 6000$ ; comparison with Itoh.

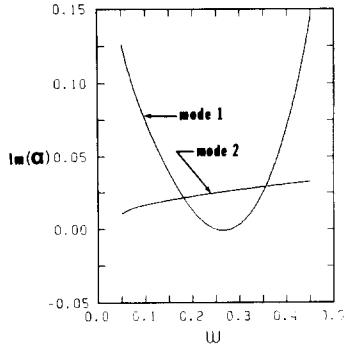


FIG. 2. Imaginary part of eigenvalue for first two modes for  $R = 6000$ ; comparison with Itoh.

travelling at the centerline velocity whereas the first mode has a phase velocity which is much lower. Since only the first mode is unstable it is not surprising that this mode was identified in the experiments of Nishioka *et al.* [16]. In the range of frequencies considered in the experiments, the first mode was the least stable. However, it is important to note that with the globally convergent Bernoulli method the first  $m$  eigenvalues may be obtained without the initial guess required in local iteration methods.

At  $R = 6000$ , the least stable eigenvalue occurs at  $\omega \sim 0.26$ . For this set of parameters, the first ten members of the eigenvalue spectrum are shown plotted in alpha space in Fig. 3. This subset of the spectrum is also shown plotted in  $C$ -space in Fig. 4.  $C$  is defined as  $C = \omega\alpha^*/|\alpha|^2$ , where the asterisk denotes complex conjugation. The numerical values for these eigenvalues are listed in Table IV. They were obtained using the Bernoulli method, and verified using the companion matrix method and the local method.

The cubically convergent local method described in Section 2.3 may also be used in stability calculations. An approximate root may be obtained using the Bernoulli method for a low order matrix and then refined using the local method on a higher order matrix. If the right eigenvector  $\mathbf{a}_i$  is defined as corresponding to the eigenvalue  $\alpha_i$  such that

$$\mathbf{D}_4(\alpha_i) \mathbf{a}_i = 0 \tag{55}$$

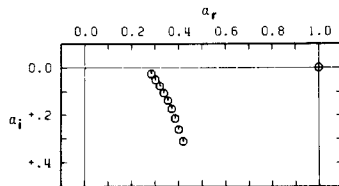


FIG. 3. The first ten members of the eigenvalue spectrum plotted in alpha space for  $\omega = 0.26$  and  $R = 6000$ .

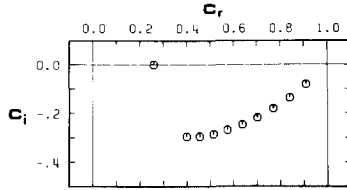


FIG. 4. The first ten members of the eigenvalue spectrum plotted in  $C$ -space for  $\omega = 0.26$  and  $R = 6000$ .

then an inverse iteration (28) may be used to determine  $\mathbf{a}_i$ . As the inverse of  $\mathbf{D}_4(\alpha_i)$  is available, from the local eigenvalue iteration, the eigenvector iteration is accomplished in  $O(N^2)$  operations.

As an example of the application of these techniques the critical neutral point is considered again. With  $R = 5772$  and  $\omega = 0.26943$  the initial guess is taken from Table III for  $N = 11$ . After 3 iterations using the local method with  $N = 40$ , the eigenvalue is given by,  $\alpha = (1.020556, 9.742(-07))$ . The associated eigenvector is plotted in Fig. 5 and its derivative is given in Fig. 6. The magnitude of the derivative gives the distribution across the channel of the axial velocity perturbations. These results agree with those of Itoh [15, Fig. 5] though Itoh found the critical Reynolds number to be 5771 with  $\omega = 0.270$ .

At a given Reynolds number, the frequency for the most unstable mode is determined precisely by the frequency for which the imaginary part of  $\partial\alpha/\partial\omega$  is zero. The value of  $\partial\alpha/\partial\omega$  may be determined by first defining the left eigenvector  $\mathbf{b}_i$  such that

$$\mathbf{b}_i^H \mathbf{D}_4(\alpha_i) = 0, \tag{56}$$

TABLE IV

The First Ten Members of the Eigenvalue Spectrum for Spatial Stability of Plane Poiseuille Flow Obtained Using the Bernoulli Method:  $R = 6000$  and  $\omega = 0.26$

Number	Eigenvalue
1	1.00047 - $i$ 0.00086
2	0.28323 + $i$ 0.02538
3	0.30165 + $i$ 0.04886
4	0.31976 + $i$ 0.07532
5	0.33745 + $i$ 0.10492
6	0.35456 + $i$ 0.13782
7	0.37090 + $i$ 0.17425
8	0.38629 + $i$ 0.21480
9	0.40156 + $i$ 0.26063
10	0.42050 + $i$ 0.31175



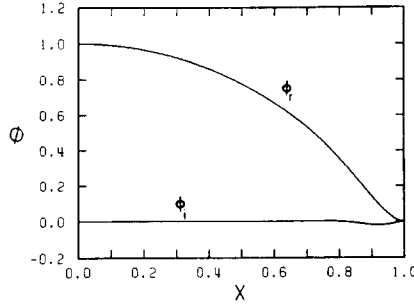


FIG. 5. Eigenfunction for  $\omega = 0.26943$  and  $R = 5772$ ; real and imaginary parts.

where  $H$  denotes the complex conjugate transpose. However,  $\mathbf{b}_i$  also satisfies

$$\mathbf{D}_4^H(\alpha_i) \mathbf{b}_i = 0 \tag{57}$$

and  $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$ . Thus, the complex conjugate transpose of the available inverse  $\mathbf{D}_4^{-1}(\alpha_i)$ , may be used in the inverse iteration (28) to compute  $\mathbf{b}_i$  in  $O(N^2)$  operations. Now the derivative of (55) with respect to  $\omega$  is

$$\frac{\partial \mathbf{D}_4}{\partial \omega} \mathbf{a}_i + \mathbf{D}_4 \frac{\partial \mathbf{a}_i}{\partial \omega} = \mathbf{0}. \tag{58}$$

Multiplying (58) by  $\mathbf{b}_i^H$  causes the second term to vanish from (56). Then if the expression (45) is used for  $\mathbf{D}_4(\alpha)$ , and noting that only  $\mathbf{C}_2$  and  $\mathbf{C}_4$  depend on  $\omega$ , the following result is obtained

$$\frac{\partial \alpha_i}{\partial \omega} = \frac{-\mathbf{b}_i^H \{ (\partial \mathbf{C}_2 / \partial \omega) \alpha_i^2 + (\partial \mathbf{C}_4 / \partial \omega) \} \mathbf{a}_i}{\mathbf{b}_i^H \mathbf{D}_4^{(1)}(\alpha_i) \mathbf{a}_i}. \tag{59}$$

The derivative  $\mathbf{D}_4^{(1)}(\alpha_i)$  is available from the local iteration method of Section 2.3. The expression (59) may be evaluated in  $O(N^2)$  operations.

As an example of the evaluation of (59) Fig. 7 shows the real and imaginary parts of  $\partial \alpha / \partial \omega$  as a function of frequency for  $R = 6000$ . This corresponds to the conditions shown in Figs. 1 and 2. The frequency for which the imaginary part of  $\partial \alpha / \partial \omega$  is zero gives the most unstable frequency for this mode. Fig. 8 shows this frequency, for the

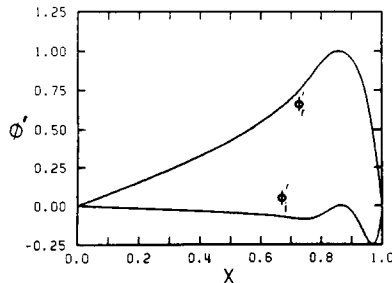


FIG. 6. Derivative of eigenfunction for  $\omega = 0.26943$  and  $R = 5772$ ; real and imaginary parts.

most unstable mode, at various values of Reynolds number  $R$ .  $\partial\alpha/\partial\omega$  is also useful for determining points on the curve of neutral stability. For example, if we seek the values of  $\omega$  and  $\alpha_r$ , at a particular value of  $R$ , which results in  $\alpha_i \rightarrow 0$ , the following algorithm can be used. Given an initial estimate for  $\omega$  and  $\alpha_r$ , a correction can be found from

$$\omega_{K+1} = \omega_K + \Delta\omega \tag{60}$$

$$(\alpha_r)_{K+1} = (\alpha_r)_K + \Delta\omega \operatorname{Re} \left[ \frac{\partial\alpha}{\partial\omega} \right]_K \tag{61}$$

$$\Delta\omega = -(\alpha_i)_K / \operatorname{Im} \left[ \frac{\partial\alpha}{\partial\omega} \right]_K \tag{62}$$

#### 4. DISCUSSION

We have not done a detailed comparison of computer times, but some estimates are given, since this is the case of most interest. Since the QR algorithm has an operation count which is of  $O(N^3)$ , we can expect the companion matrix method to require about 64 times more computer time because of its enlarged size. The Bernoulli method requires 3 or 4 matrix multiplies and 1 matrix inversion per iteration. Therefore, its operation count is  $O(N^3)$  per iteration. Using a regular dense matrix multiply and a fully pivoted Gauss–Jordan inversion routine, we found that 25 Bernoulli iterations is about equivalent to the companion matrix method, when the companion matrix has been pre-conditioned by inverting  $C_0$  and multiplying through. The Bernoulli method is clearly faster. It also has the potential for significant increases in efficiency with the aid of currently available fast matrix inversion and multiply routines.

Regarding the convergence of the Bernoulli method for solvents of matrix polynomials, a proof is given in Ref. [5, p. 126] for a particular case. A Bernoulli

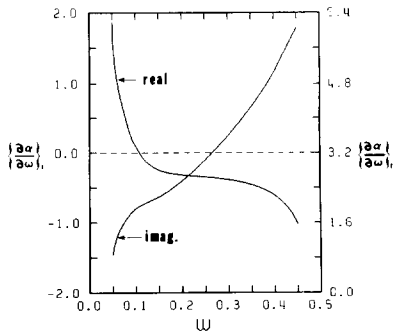


FIG. 7. Variation of  $\partial\alpha/\partial\omega$  versus frequency for  $R = 6000$ ; real and imaginary parts.

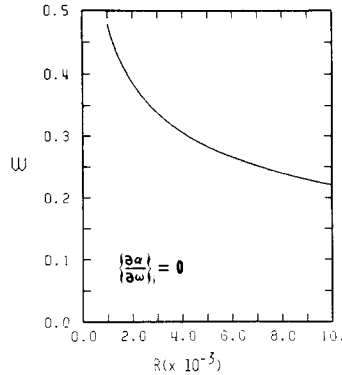


FIG. 8. Variation of values of  $\omega$  as a function of  $R$  for which the imaginary part of  $\partial\alpha/\partial\omega = 0$ .

sequence is derived for a monic matrix polynomial which converges to the dominant solvent. A monic matrix polynomial is one in which the leading coefficient matrix is the identity matrix. It appears that this proof can be extended to the case of the minimal solvent, and the numerical evidence seems to support this. However, there is a further question regarding convergence for the case where the matrix polynomial is non-monic and also the case where it is non-monic and the leading coefficient matrix is singular. We are currently studying this and hope to report in the future.

For the model equation and the Orr–Sommerfeld equation considered in the previous sections it was only necessary to remove the boundary conditions to make the leading coefficient matrix nonsingular. This matrix could then be inverted and a monic lambda matrix could be formed. However, this is not the case in general. For example, consider the model problem, which is a slight modification of (2),

$$\frac{d^2\phi}{dx^2} - 2\alpha^2 \frac{d\phi}{dx} + \alpha\omega\phi = 0, \tag{63}$$

$$\phi(\pm 1) = 0, \tag{64}$$

for which the exact eigenvalue relationship is

$$\alpha^4 - \alpha\omega + (n\pi/2)^2 = 0, \quad n = 1, 2, \dots \tag{65}$$

Following the procedures developed above (63) is integrated twice, solutions of the form (6) are sought, and the tau method is used. This leads to  $N - 1$  equations

$$a_n - \frac{\alpha^2}{n} (a_{n-1} - a_{n+1}) + \frac{\alpha\omega}{4n} \left[ \frac{a_{n-2}}{(n-1)} - \frac{2na_n}{(n^2-1)} + \frac{a_{n+2}}{(n+1)} \right] = 0, \quad n = 2, \dots, N, \tag{66}$$

and the two boundary conditions (11). This results in a lambda matrix of degree two

$$D_2(\alpha) = C_0\alpha^2 + C_1\alpha + C_2. \tag{67}$$

The lowest two rows of  $C_0$  and  $C_1$  are zero and the lowest two rows of  $C_2$  contain the boundary conditions. The lowest two rows of  $C_2$  are eliminated using column operations and the  $2 \times 2$  submatrix is formed. As before the problem is reduced to order  $N - 1$ . However, in this case, the matrix  $C_0$  is still singular. Thus, the system contains an “infinite” eigenvalue, since  $\det C_0$  is the leading coefficient of the characteristic equation formed from (67). An alternative is to convert to the generalized eigenvalue problem (16). However, this requires considerably more computation time than the ordinary eigenvalue problem. For quartic or higher order nonlinear eigenvalue problems, the penalty would be even greater. In practice, the solution of the generalized eigenvalue problem (46), using the rank iteration scheme of Gary and Helgason [2], was found to require about thirty times more computer time than inverting  $C_0$  and solving (47). Thus, there is a tremendous saving if the lambda matrix can be made monic. It should be noted that this is not a requirement for the Bernoulli method of matrix factorization.

To make the leading coefficient matrix nonsingular the following transformation may be used:

$$\lambda = (z_1 + \alpha)/(z_2 - \alpha), \tag{68}$$

This results in a new lambda matrix

$$\tilde{D}_2(\lambda) = A_0\lambda^2 + A_1\lambda + A_2, \tag{69}$$

where

$$\begin{aligned} A_0 &= C_0z_2^2 + C_1z_2 + C_2 \\ A_1 &= -2C_0z_1z_2 + C_1(z_2 - z_1) + 2C_2 \\ A_2 &= C_0z_1^2 - C_1z_1 + C_2. \end{aligned} \tag{70}$$

If  $z_2$  is not an eigenvalue  $A_0$  may now be inverted. For example, consider the calculations for  $N = 10$ ,  $\omega = 1$  and  $z_1 = z_2 = 1$ . The results are shown in Table V. The exact values are given by (65). Since the “infinite” eigenvalue is only concealed

TABLE V  
Eigenvalues for Problem (64) in Which  $C_0$  is Singular Obtained Using (65)  
with  $z_1 = z_2 = 1$ ,  $\omega = 1$ ,  $N = 10$

Eigenvalue Numbers $i$	Eigenvalue $\alpha_i$	Error
1	(0.893180, 0.719632)	5.26(-07)
3	(-0.893180, 1.038110)	2.37(-06)
5	(1.254089, 1.172177)	5.65(-04)
⋮	⋮	⋮
18	(1.926(+15), 2.019(+14))	1.94(+15)

by the transformation (68), it still appears as the very large eigenvalue in the table. This large eigenvalue appears to have no effect on the other smaller eigenvalues.

The use of the companion matrix method for the Orr–Sommerfeld discretization leads to the appearance of “spurious” eigenvalues. A “spurious” eigenvalue is one which is not an eigenvalue of the Orr–Sommerfeld operator. It may be attributed to either the Chebyshev–tau method [17, p. 114] or the linear algebraic method of solution of the matrix problem. These eigenvalues may be either unstable ( $\text{Im}(\alpha) < 0$ ) or stable ( $\text{Im}(\alpha) > 0$ ). Spurious stable modes can be tolerated since even true stable eigenvalues are usually of little physical importance. However, unstable eigen-solutions are of considerable importance. For the present calculations of channel flow both the stable and unstable spurious eigenvalues have such large magnitudes that they are easily distinguished from the true eigenvalues. However, this may not be the case in all problems.

Some of the spurious modes may be attributed to the disparity in the norms of the coefficient matrices at high Reynolds numbers. In the Orr–Sommerfeld problem all the elements of  $\mathbf{C}_0$  are less than unity while the elements of  $\mathbf{C}_1$  through  $\mathbf{C}_4$  are of the order of the Reynolds number. At high Reynolds numbers the matrix  $\mathbf{C}_0$  becomes negligible. The elements of  $\mathbf{C}_0$  form the coefficients of the terms proportional to the highest powers of  $\alpha$  in the scalar characteristic equation for (45). Thus,  $m$  very large eigenvalues should be expected where  $m$  is the order of (45). Since  $\mathbf{C}_0$  is negligibly small the eigenvalue problem could be reduced to a cubic nonlinear eigenvalue problem. The spectrum of eigenvalues in this case is found to be the same as the spectrum for the fourth-order problem without the  $m$  largest eigenvalues. For low-order matrices the eigenvalues contributed by  $\mathbf{C}_0$  are all stable but for larger matrices some are unstable. The use of the Bernoulli method filters out the spurious eigenvalues when solving for the minimal solvent. Since the spurious eigenvalues should have, as a rule, large magnitude, the method of matrix factorization offers a convenient method for their elimination.

As in the scalar case, the matrix Bernoulli method has less than quadratic convergence. For improving convergence Newton’s method might be considered. However, in the case of matrix polynomials, its implementation is not straightforward. Consider the quadratic matrix polynomial

$$\mathbf{D}_2(\mathbf{Y}) = \mathbf{Y}^2 + \mathbf{C}_1 \mathbf{Y} + \mathbf{C}_2 = \mathbf{0}. \quad (71)$$

An improved estimate is sought of the form

$$\mathbf{Y}_{i+1} = \mathbf{Y}_i + \Delta_i. \quad (72)$$

This results in a linearized equation for  $\Delta_i$  of the form

$$(\mathbf{Y}_i + \mathbf{C}_1) \Delta_i + \Delta_i \mathbf{Y}_i = -\mathbf{D}_2(\mathbf{Y}_i). \quad (73)$$

This reduces to the more common form of Newton’s method if  $\Delta_i$  and  $\mathbf{Y}_i$  commute. Matrix equations of the form (73) are common in control theory and algorithms for

their solution are available [18]. However, the  $QR$  algorithm is used to triangularize the known coefficient matrices and the solution is obtained by back substitution. This results in a high operation count per iteration. Thus, it is not clear that there are any benefits to be gained from this implementation of Newton's method over the simpler implementation of Bernoulli's method. The use of Newton's method for quadratic matrix polynomials is discussed by Lancaster and Rakne [19] and an algorithm has been developed by Davis [20, 21]. Attempts at extending Newton's method to cubic and higher degree matrix polynomials have not been fruitful. For example, if Newton's method is applied to a monic cubic matrix polynomial a solution is required to

$$(\mathbf{Y}_i^2 + \mathbf{C}_i \mathbf{Y}_i + \mathbf{C}_2) \Delta_i + \Delta_i \mathbf{Y}_i^2 + (\mathbf{Y}_i + \mathbf{C}_1) \Delta_i \mathbf{Y}_i + \Delta_i \mathbf{Y}_i + \Delta_i \mathbf{Y}_i \Delta_i = -D_3(\mathbf{Y}_i). \quad (74)$$

It is not clear whether  $\Delta_i \mathbf{Y}_i \Delta_i$  may be neglected. Even if it is, the method for the solution of the remaining matrix equation for  $\Delta_i$  is also unclear. Obviously, further complications arise for problems of higher degree.

It has been suggested that the method of Gary and Helgason for the generalized eigenvalue problem has been made obsolete by the introduction of the  $QZ$  algorithm of Moler and Stewart. This is only partially true. Although this algorithm is widely available through the IMSL package, it is for matrices with *real* entries only, although it can still determine complex eigenvalues. Since the Orr-Sommerfeld equation has complex coefficients except in the limiting case of zero frequency, we require a method which handles complex matrices. To avoid modifying the  $QZ$  algorithm, we used the Gary/Helgason algorithm which is publicly available through NCAR. However, a complex version of  $QZ$  has recently been developed by George Davis and is now publicly available [20], and IMSL version 9, put out in the summer of 1982, also contains a complex version of  $QZ$  called EIGZC. It is likely that this will now become the algorithm of choice for the complex generalized eigenvalue problem.

The authors' original interest in these nonlinear eigenvalue problems grew from an interest in the applications of linear hydrodynamic stability theory. Extensions of the present analysis to consider the stability of unbounded shear flows and the influence of nonparallel mean flows are being examined.

#### APPENDIX

The coefficients in Eq. (37) are defined as

$$\tilde{A}_n = \frac{a_{n-1}}{8n(2n-1)} - \frac{a_n}{2(2n-1)(2n+1)} + \frac{a_{n+1}}{8n(2n+1)}.$$

$$\tilde{B}_n = \frac{a_{n-2}}{32n(2n-1)(2n-2)(2n-3)} - \frac{a_{n-1}}{8n(2n-3)(2n-1)(2n+1)}$$

$$\begin{aligned}
& + \frac{a_{n+2}}{32n(2n+1)(2n+2)(2n+3)}. \\
\tilde{C}_n &= \frac{a_{n-2}}{32n(2n-1)(2n-2)} - \frac{a_{n-1}}{16n(2n-1)(2n+1)} \\
& - \frac{a_n}{8(2n-1)(2n-2)(2n+1)(2n+2)} + \frac{a_{n+1}}{16n(2n-1)(2n+1)} \\
& - \frac{a_{n+2}}{32n(2n+1)(2n+2)}. \\
\tilde{D}_n &= \frac{a_{n-2}}{16n(2n-1)} - \frac{(a_{n-1} - a_n + a_{n+1})}{4(2n-1)(2n+1)} + \frac{a_{n+2}}{16n(2n+1)}. \\
\tilde{E}_n &= \frac{a_{n-3}}{64n(2n-3)(2n-2)(2n-1)} - \frac{a_{n-2}}{16n(2n-1)(2n-3)(2n+2)} \\
& + \left\{ \frac{(14n-1)}{64n(2n-3)(2n-1)(2n+1)(2n+2)} + \frac{\delta_{2n}}{768} \right\} a_{n-1} \\
& - \frac{a_n}{4(2n-1)(2n+1)(2n-3)(2n+3)} + \frac{(14n+1)a_{n+1}}{64n(2n-2)(2n-1)(2n+1)(2n+3)} \\
& - \frac{a_{n+2}}{16n(2n-1)(2n+1)(2n+3)} + \frac{a_{n+3}}{64n(2n+1)(2n+2)(2n+3)}.
\end{aligned}$$

$\delta_{ij}$  is the kronecker delta function defined such that

$$\delta_{2n} = \begin{cases} 1 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

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